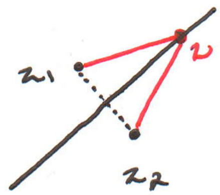
Math 742 H.W. # 1

1. (Stein 1-1) Describe geometrically the set of points z in the complex plane defined by the following relations:

(a) $|z-z_1| = |z-z_2|$ where $z_1, z_2 \in \mathbb{C}$.

Solution:



The set is a line orthogonal to the line segment $[z_1, z_2]$ through the mid-pt.

(b) $\frac{1}{z} = \bar{z}$

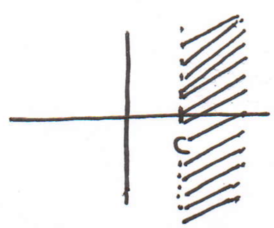
Solution: $1 = \bar{z}z = |z|^2$ is the unit circle.

(c) $Re(z) = 3$

Solution: This is a vertical line $x = 3$

(d) $Re(z) > c$ where $c \in \mathbb{R}$

Solution: This is the half-plane to the right of $x = c$



(e) $Re(az+b) > 0$

Solution: This is the half-plane that the map $z \mapsto az+b$ moves into the right half plane.

(f) $|z| = Re(z) + 1$

Solution: $|z| = \sqrt{x^2+y^2} = Re(z) + 1 = x + 1$. This equation reduces to $y^2 = 2x + 1$, which is a parabola.

(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$

Solution: This is the horizontal line $y = c$.

2. (Stein 1-8) Suppose $\mathcal{U}$ and V are open sets in the complex plane. Prove that if $f: \mathcal{U} \mapsto V$ and $g: V \mapsto \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$, then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z} \quad \text{and} \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

This is the complex version of the chain rule.

Solution: We use differential notation to prove both results simultaneously:

(i) First note that $df = f_x dx + f_y dy = f_x \frac{dz + d\bar{z}}{2} + f_y \frac{dz - d\bar{z}}{2i} =$

$= \frac{1}{2}\left(f_x + \frac{1}{i} f_y\right) dz + \frac{1}{2}\left(f_x - \frac{1}{i} f_y\right) d\bar{z} = f_z dz + f_{\bar{z}} d\bar{z}$.

(ii) Thus $d(g \circ f) = (g_\xi \circ f) df + (g_{\bar{\xi}} \circ f) d\bar{f} =$

$= (g_\xi \circ f)(f_z dz + f_{\bar{z}} d\bar{z}) + (g_{\bar{\xi}} \circ f)(\bar{f}_z dz + \bar{f}_{\bar{z}} d\bar{z}) =$

$$= \underbrace{\left[(g_\xi \circ f) f_z + (g_{\bar{\xi}} \circ f)\bar{f}_z\right]}_{\frac{\partial h}{\partial z}} dz + \underbrace{\left[(g_\xi \circ f) f_{\bar{z}} + (g_{\bar{\xi}} \circ f)\bar{f}_{\bar{z}}\right]}_{\frac{\partial h}{\partial \bar{z}}} d\bar{z}.$$

3. (Stein 1-9) Show that in polar coordinates, the Cauchy-Riemann equations take the form $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$.

Use these equations to show that the logarithm function defined by $\log z = \log r + i\theta$ (where $z = re^{i\theta}$ $-\pi < \theta < \pi$) is holomorphic in the region $r>0$ and $-\pi<\theta<\pi$

Solution: Let $f(z) = f(r,\theta) = u(r,\theta) + iv(r,\theta)$. If $f(z)$ is complex differentiable at $z_0 = (r_0, \theta_0)$ then

(i) $$f'(z_0) = \lim_{r \mapsto r_0} \frac{f(r,\theta_0) - f(r_0,\theta_0)}{re^{i\theta_0} - r_0 e^{i\theta_0}} = \frac{1}{e^{i\theta_0}} \frac{\partial f}{\partial r}(z_0) =$$

$$= \frac{1}{e^{i\theta_0}} \left(\frac{\partial u}{\partial r}(z_0) + i \frac{\partial v}{\partial r}(z_0) \right)$$

(ii) $$f'(z_0) = \lim_{\theta \mapsto \theta_0} \frac{f(r_0,\theta) - f(r_0,\theta_0)}{r_0 e^{i\theta} - r_0 e^{i\theta_0}} = \lim_{\theta \mapsto \theta_0} \frac{1}{r_0} \frac{f(r_0,\theta) - f(r_0,\theta_0)}{\theta - \theta_0} \cdot$$

$$\cdot \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}} = \frac{1}{r_0} \frac{\partial f}{\partial \theta}(z_0) \cdot \frac{1}{i e^{i\theta_0}} = \frac{1}{i z_0} \frac{\partial f}{\partial \theta}(z_0) =$$

$$= \frac{1}{i z_0} \left(\frac{\partial u}{\partial \theta}(z_0) + i \frac{\partial v}{\partial \theta}(z_0) \right)$$

(iii) It follows that $\frac{r_0}{z_0} \left(\frac{\partial u}{\partial r}(z_0) + i \frac{\partial v}{\partial r}(z_0) \right) =$

$= \frac{1}{i z_0} \left(\frac{\partial u}{\partial \theta}(z_0) + i \frac{\partial v}{\partial \theta}(z_0) \right)$. Hence $r_0 \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$ and

$r_0 \frac{\partial v}{\partial r} = - \frac{\partial u}{\partial \theta}$, which is equivalent to the statement that we wished to prove.

Finally, observe that $\log z$ has continuous polar-form partial derivatives that satisfy the Cauchy-Riemann equations. Therefore $\log z$ is holomorphic in the region. Furthermore $\frac{d}{dz}(\log z) = \frac{1}{iz}\left(\frac{\partial \log r}{\partial \theta} + i \frac{\partial \theta}{\partial \theta} \right)$

$= \frac{1}{z}$.

4. (Stein 1-10) Show that $4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta$ where Δ is the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Solution: $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$

Then $4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) =$

$= \frac{\partial^2}{\partial x^2} - i\frac{\partial^2}{\partial x \partial y} + i\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta$ by equality of mixed partials.

5. (Stein 1-11) Prove that if f is holomorphic in the open set Ω, then the real and imaginary parts of f are harmonic; that is, their Laplacian is zero.

Solution: Let $f = u + iv$. Then $\Delta(f) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}(f) = 0$ because $\frac{\partial}{\partial \bar{z}}(f) = 0$. This, of course, implies that $(u_{xx} + u_{yy}) + i(v_{xx} + v_{yy}) = 0$

$\Rightarrow \Delta(u) = \Delta(v) = 0$.

6. (Stein 1-12) Consider the function defined by $f(x+iy) = \sqrt{|x||y|}$ whenever $x, y \in \mathbb{R}$. Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Solution: $$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{\sqrt{|h|\cdot 0} - \sqrt{0\cdot 0}}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{\sqrt{0\cdot|k|} - \sqrt{0\cdot 0}}{k} = 0$$

Hence $i\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0)$, which means that the Cauchy-Riemann equations hold.

However, f is not holomorphic at 0 because $\lim\limits_{h \mapsto 0} \frac{f(h)-f(0)}{h} =$
$= f_x(0) = 0$ if $h \in \mathbb{R}$, while

$$\lim_{\substack{h+ik \mapsto 0 \\ h=k}} \frac{f(h+ik)-f(0)}{h+ik} = \lim_{h \mapsto 0} \frac{|h|}{h(1+i)} = \frac{\pm 1}{1+i} \neq 0.$$

7. (Stein 1-13) Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases one can conclude that f is constant:

(a) $Re(f)$ is constant

Solution: $f = u + iv$ satisfies $f_x = i v_x$ and $f_y = i v_y$. By the Cauchy-Riemann equations, $i f_x = f_y$, $-v_x = i v_y \Rightarrow v_x = v_y = 0$

So $v = Im(f)$ is constant as well.

(b) $Im(f)$ is constant.

Solution: let $f = u + iv$ and $g = -if$. Then $Re(g) = Im(f)$ is constant. By (a), g must be constant and so is f.

(c) $|f|$ is constant.

Solution: In polar form $f(x,y) = Re^{i\varphi(x,y)}$ where $R = |f|$ is constant. By the Cauchy-Riemann equation $i f_x = f_y$ so

$i\left(iRe^{i\varphi} \frac{\partial \varphi}{\partial x}\right) = iRe^{i\varphi} \frac{\partial \varphi}{\partial y}$ whence $-\frac{\partial \varphi}{\partial x} = i \frac{\partial \varphi}{\partial y} \Rightarrow$

$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0$. Thus $\varphi(x,y) = \varphi$ is also constant.